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THE DOUBLE BUBBLE PROBLEM IN SPHERICAL AND HYPERBOLIC SPACE

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Abstract

We prove that the standard double bubble is the least-area way to enclose and separate two regions of equal volume in H3, and in S3 when the exterior is at least ten percent of S3.That is standard double bubble is the area-minimizing partition of spheres of any dimension where the volumes differ by at most 4%.

Key Words Bubble,Projection,Gauss Space,Volume,Minimize

Introduction

the Double Bubble Conjecture, which says that the area-minimizing way to enclose and separate two given volumes in R3 is a standard double bubble, defined as three spherical caps meeting in threes at 120 degrees. There have been partial extensions to Rn, the sphere S3 , and hyperbolic space H3 ([Rei],[CF], [CH]). We provide extensions to S n and Gauss space Gm for the case where each volume is approximately 1/3 of the total volume of the space.

Single and Double Bubbles in Gauss Space

Gauss space Gm, of much interest to probabilists is Rm endowed with density $(2\pi)^{-m/2} e^{-x^2/2}$, with total volume 1. Given any three positive volumes V1 + V2 + V3 = 1, there exists a standard Y, unique up to symmetries of Gm, consisting of three halfhyperplanes meeting at 120 degrees, that partitions Gm into three regions of those prescribed volumes (Proposition 2.11). Note that the center m − 2 dimensional meeting plane of a Y passes through the origin if and only if the three volumes are equal.

Theorem: A least-area enclosure of two equal volumes in S3 which add up to at most 90 percent of the total volume of S3 must be the (unique) standard double bubble

Theorem : The standard Y is an area-minimizing partition of Gm for three given positive volumes V1 + V2 + V3 = 1, under the hypothesis that the standard double bubble for the same volume fractions is area minimizing in S n for infinitely many n. Section 3 proves that the standard double bubble is area minimizing in S n for all n for nearly equal volumes. Using Brakke's Surface Evolver, provides corroborating computer evidence that a standard Y is area minimizing for volumes (1/3, 1/3, 1/3) (as we prove) and for volumes (1/3, 1/10, 17/30) (which is outside our region of proof).

This evidence makes it likely that the standard Y is minimizing for all volumes .It is worthwhile to note that while Section 3 proves a unique area-minimizing partition for certain volumes in S n, the uniqueness of the corresponding partition in Gm is an open question. The Proof. The main idea comes from Mehler's 1856 observation that Gauss space is weakly the limit of projections of suitably normalized high-dimensional spheres as used by Borell [Bo] and Sudakov-Tsirel'son [ST] to prove hyperplanes optimal single bubbles in Gm. The proof requires stronger convergence of perimeter for fixed volumes including a technical Lemma ,how to repair small volume discrepancies at low area cost.

Double Bubbles in S^n

We prove the Double Bubble Conjecture in the sphere S^n for volume fractions near 1/3:

The Double Bubble Conjecture in Gauss Space Our main result proves that the standard Y is an area-minimizing partition of Gauss space G^m into three prescribed volumes, as long as the standard double bubble is area minimizing in high-dimensional spheres for the same volume fractions. It provides the main idea that Gauss space is the limit of projections of high-dimensional spheres.

Area and Volume Convergence in Gauss Space

Gauss space G^m is R^m endowed with Gaussian density $f(x) = (2\pi)^{-m/2} e^{-x^2/2}$. We recall that volume and area in G^m and in manifolds with density in general, are given by integrating the density with respect to the Euclidean volume and area (Hausdorff measure). Propositions 2.1 and 2.2 show that this density can be obtained by projecting high-dimensional spheres $S^n(\sqrt{n})$ of radius $\sqrt{n}$ into R^m. It shows that the areas of the inverse projections in S^n of a hypersurface in G^m converge to the area of the hypersurface as n approaches infinity. The following proposition, often attributed to Poincar´e, is due to Mehler

Proposition (Mehler, 1856). Gaussian measure on R^m is obtained as the limit as n approaches infinity of (orthogonal) projections P_n of uniform probability density on $S^n(\sqrt{n}) \subset R^{n+1}$ to R^m.

Proof : Observe that G^m is the projection of G^n onto its first m coordinates. The coordinates $X_1, \ldots, X_n$ of G^n are Gaussian random variables. By the Central Limit Theorem, the distance to the origin $\sqrt{X_1^2 + \cdots + X_n^2}$ converges to $\sqrt{n}$ as $n \to \infty$. By the spherical symmetry of G^n, the volume of G^n concentrates uniformly on $S^{n-1}(\sqrt{n})$ for large n. Therefore, G^m can be obtained as the limit as $n \to \infty$ of projections P_n of the uniform probability density on $S^{n-1}(\sqrt{n})$ or $S^n(\sqrt{n})$ to R^m. Furthermore, the induced density converges pointwise, not just in measure, to the Gaussian density.

Proposition :The projections P_n of the uniform probability density on $S^n(\sqrt{n})$ to R^m for fixed m converge pointwise to Gaussian density on R^m.

Proof: For a fixed m, let $f_n(x)$ be the density function resulting from projecting the uniform probability measure on $S^n(\sqrt{n})$ to R^m. By symmetry, we may assume that x lies on the nonnegative first coordinate axis. Let θ be the angle between $S^n(\sqrt{n})$ and R^m; then $\sin\theta = x/\sqrt{n}$. A volume element dV of S^n is projected to a weighted volume element $dV \cos\theta$ on R^m. Therefore, a volume element dV on R^m has weighted volume $dV/(\cos\theta)$. Observe that $P_n^{-1}(x)$ is the $(n-m)$-sphere with radius $(\cos\theta)\sqrt{n}$ and thus has area proportional to $(\cos\theta)^{n-m-1}$. Therefore,

$$f_n(x) = c_n(2\pi)^{-m/2} (\cos\theta)^{n-m-2}$$

where c_n is some normalization constant. Since $\cos^2\theta = 1 - \sin^2\theta = 1 - x^2/n$, $f_n(x) = c_n(2\pi)^{-m/2} (1 - x^2/n)^{(n-m-2)/2}$. Since $(1 - x^2/n)^{(n-m-2)/2}$ converges to $e^{-x^2/2}$ as $n \to \infty$, $c_n \to 1$, because the $f_n(x)$ converge to $f(x)$, the Gaussian density, in measure by above Proposition .

The Y and the Standard Y

This section introduces the standard Y and its properties. Proposition 2.12 proves that the area of the standard double bubble on $S^n(\sqrt{n})$ converges to the area of the standard Y in G^m of the same volume fractions as n grows large. Definition 1. A Y in Gm consists

Definition. A Y in G^m consists of three half-hyperplanes meeting along an $(m-2)$-dimensional plane. Let β_i denote the angles between the hyperplanes. A standard Y is a Y with each $\beta_i = 2\pi/3$. Let θ denote the angle with the horizontal as in Figure 1. By O(m) symmetry, we may assume that the $(m-2)$-dimensional plane lies normal to the x-y plane and contains the nonnegative x-axis. The Y partitions G^m into three volumes V_1, V_2, and V_3. The volumes partitioned by the standard Y satisfy $V_1 \geq V_2 \geq V_3$ as in Figure 1. Let $V_i(x, \theta, \beta_1, \beta_2)$ and $A_i(x, \theta, \beta_1, \beta_2)$ denote the volume and area, respectively, of the i th region as functions of x, θ, β_1, and β_2

Proposition : For two prescribed volumes v, w (with $v + w < \mathrm{vol}(S^n)$), there is a unique standard double bubble in S^n (up to isometries) consisting of three spherical caps meeting at 120 degrees that encloses volumes v and w.

For two prescribed volumes v, w, there is a unique standard double bubble in Hn (up to isometries) consisting of three spherical caps meeting at 120 degrees that encloses volumes v and w. The outer two caps are pieces of spheres, and the middle cap may be any spherical surface.

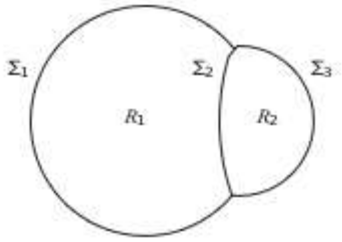

FigurE 2.1. Construction of a standard double bubble from three spherical caps.

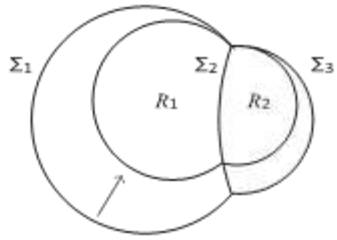

FigurE 2.2. Increasing the curvature of Σ1 while keeping the curvature Σ2 fixed increases the curvature of Σ3. The volumes of R1 and R2 both de- crease.

Proof. Masters [12, Theorem 2.2] proved the existence and uniqueness of the stan- dard double bubble in S2; this result generalizes directly to Sn merely by considering spherical caps instead of circles. We use similar methods for Hn. The main idea of the proof is to parameterize double bubbles by the mean curvatures of one of the outer caps and the middle cap.

Consider two mean curvatures (sums of principal curvatures) H1 ∈ (n □ 1, □) and H2 ∈ [0, □). Draw two spherical caps Σ1, Σ2 with these mean curvatures, meeting at 120 degrees, so that Σ1 has positive mean curvature when considered from the side the angle is measured on and Σ2 has negative mean curvature when considered from this side. It is obvious that the caps must meet up, since Σ1 is a portion of a sphere (because H1 > n□ 1). Denote the enclosed region R1. Complete this figure to a double bubble with a third spherical cap Σ3 that meets the other two at 120 degrees at their boundary, enclosing a second region R2. Note that Σ2 will necessarily be the middle cap. (See Figure 2.1.) Obviously there is at most one way to do this. To see that this can always be done, note that if H2 is equal to zero, then Σ1 and Σ3 are identical. As we increase H2 with H1 fixed, the mean curvature of Σ3 increases, as shown in Figure 2.3.Thus Σ3 has mean curvature greater than or equal to H1 and is thus a portion of a sphere, so there is no problem with surfaces going off to infinity without meeting up. Let V1 be the volume of R1 and V2 be the volume of R2. Define a map F : (n□ 1, □)× [0, □) → □(x, y) ∈ R>0 ×R>0 □ x □ y□ such that F(H1,H2) □ (V1,V2). As can be seen in Figures 2.2 and 2.3, with H2 fixed, as H1 increases, both V1 and V2 decrease. With H1

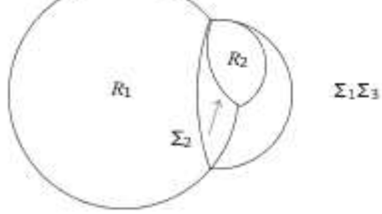

FigurE 2.3. Increasing the curvature of Σ2 while keeping the curvature of Σ1 fixed increases the curvature of Σ3. The volume of R1 increases and the volume of R2 decreases.

fixed, as H2 increases, V1 increases and V2 decreases. (Note that V2 ≤ V1, with equality only at H2 □ 0.) Thus we conclude that the map F is injective.

To show that F is surjective, we first note that the map is continuous. We now consider limiting cases. With H1 fixed, as H2 goes to zero (and Σ2 becomes a geodesic plane), the two volumes enclosed become equal. With H2 fixed, as H1 approaches infinity, both volumes V1 and V2 approach zero. With H1 fixed, as H2 goes to infinity, V1 approaches the volume of a sphere of mean curvature H1 and V2 goes to zero. With H2 fixed, as H1 decreases, V1 increases without bound. By continuity of F , all volumes (V1,V2) are achieved by our construction. Note that since V1 □ V2, V1 must become infinite first, which will happen when H1 □ n □ 1 and Σ1 becomes a horosphere. Thus F is surjective. In addition, each outer cap must be a sphere and not a horosphere or a hyposphere.

By construction, the total volume of the double bubble is greater than the total volume of a spherical surface with mean curvature H1, so if H1 ≤ n □ 1, the enclosed volume is infinite. Thus we achieve each pair of volumes V1, V2 with a standard double bubble and that every finite-volume standard double bubble is achieved in our construction, so we have the stated result.

Conculsion

Bubbles are nature's way of finding optimal shapes to enclose certain volumes.
Bubbles are studied in the fields of mathematics called Differential Geometry and
Calculus of variations.While it is possible to produce many bubbles through physical experiments,many of the mathematical properties of bubbles remain exclusive.Double bubbles gives the best way of enclosing two equal volumes.

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